The equations $2^N \pm 2^M \pm 2^L = z^2$

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1. Introduction

In the present paper we solve the title equations. It is easy to see that they lead either to

$$2^n \pm 2^m \pm 1 = x^2 \quad , \tag{1}$$

or to

$$2^n \pm 2^m \pm 2 = x^2 \quad . \tag{2}$$

While the examination of (2) is quite simple, as well as the resolution of $2^n \pm 2^m - 1 = x^2$, the equation

$$2^n + 2^m + 1 = x^2 \tag{3}$$

requires more calculations and the application of some deep results of Beukers [2]. This problem has been posed by professor Tijdeman, and I heard it from Tengely.

¿From a wider point of view, equations of types similar to (1) and (2) have already been investigated. Gerono [4] proved that a Mersenne-number $M_k = 2^k - 1$ cannot be a power of a natural number if $k > 1$, so the equation $2^k - 1 = x^2$ has only the solutions $(k, x) = (0, 0)$, $(1, 1)$. For another example, it can readily be verified that $2^k + 1 = x^2$ implies $(k, x) = (3, 2)$.

Ramanujan [7] conjectured that the diophantine equation

$$2^k - 7 = x^2 \tag{4}$$

has five solutions, namely $(k, x) = (3, 1)$, $(4, 3)$, $(5, 5)$, $(7, 11)$ and $(15, 181)$. His conjecture was first proved by Nagell [6]. The generalized Ramanujan-Nagell equation

$$2^k + D = x^2 \tag{5}$$

in natural numbers k and x, where $D \neq 0$ is an integer parameter, was considered by several authors. See, for example, Apéry [1], Hasse [5], Beukers [2]. Taking $D = \pm 2^M \pm 2^L$, we investigate infinitely many generalized Ramanujan-Nagell equations.

Our main result is Theorem 1. Theorems 1 and 2 have interesting consequences connected to binary recurrences (Corollary 1). Finally, a corollary of Lemma 5 states that the ratio of two distinct triangular numbers cannot be a power of 4 (Corollary 2).

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2. Results

Theorem 1. If the positive integers n, m and x with $n \geq m$ satisfy

$$2^n + 2^m + 1 = x^2 \quad , \tag{6}$$

then
(i) $(n, m, x) \in \{(2t, t+1, 2^t + 1) \mid t \in \mathbb{N},\ t \geq 1\}$ or
(ii) $(n, m, x) \in \{(5, 4, 7)\ ,\ (9, 4, 23)\}$.

Remarks.
I. Equation (6), essentially, asks for odd natural numbers x whose squares contain exactly three 1 digits with respect to the base 2. Theorem 1 says that beside the infinite set $x^2 = 101_2^2 = 11001_2$, $1001_2^2 = 1010001_2$, ..., only $x^2 = 111_2^2 = 110001_2$ and $x^2 = 10111_2^2 = 1000010001_2$ possess the property above.
II. The solutions (i) and (ii) in Theorem 1 enable to determine all $n, m \in \mathbb{Z}$, $x \in \mathbb{Q}$ satisfying (6).

Theorem 2. If the positive integers n, m and x satisfy

$$2^n - 2^m + 1 = x^2 \quad , \tag{7}$$

then
(i) $(n, m, x) \in \{(2t, t+1, 2^t - 1) \mid t \in \mathbb{N},\ t \geq 2\}$ or
(ii) $(n, m, x) \in \{(t, t, 1) \mid t \in \mathbb{N},\ t \geq 1\}$ or
(iii) $(n, m, x) \in \{(5, 3, 5)\ ,\ (7, 3, 11)\ ,\ (15, 3, 181)\}$.

Theorem 3. If the positive integers n, m and x with $n \geq m$ satisfy

$$2^n + 2^m - 1 = x^2 \quad , \tag{8}$$

then
(i) $(n, m, x) = (3, 1, 3)$.
Moreover, all the solutions of the equation

$$2^n - 2^m - 1 = x^2 \tag{9}$$

in positive integers n, m and x are given by
(ii) $(n, m, x) = (2, 1, 1)$.

One can find lots of results concerning occurrence of squares and higher powers in binary (or higher order) recurrences. See, for instance, Shorey, Tijdeman [8], Chapter 9. Corollary 1 determines all square terms in certain binary recursive sequences.

Corollary 1. (Corollary of Theorems 1 and 2.) Let d be an arbitrarily fixed natural number. Consider the binary recurrences

$$G_m = 3G_{m-1} - 2G_{m-2} \quad (m \geq 2)\ , \quad G_0 = 2^d + 2\ ,\ G_1 = 2^{d+1} + 3\ ; \tag{10}$$

$$H_m = 3H_{m-1} - 2H_{m-2} \quad (m \geq 2)\ , \quad H_0 = 2^d\ ,\ H_1 = 2^{d+1} - 1\ . \tag{11}$$

(i) The only square occurring in the recursive sequence G is G_{d+2}, except for the following two cases. If $d = 1$, then G contains three squares, namely G_0, $G_3 = G_{d+2}$ and G_4. If $d = 5$, then G_4 and $G_7 = G_{d+2}$ are the squares in G.

(ii) If d is odd, then

$$H_m = w^2 \tag{12}$$

implies $m = d + 2$. If $d > 0$ is even, then equation (12) has exactly two solutions given by $m = 0$ and $m = d + 2$, except for three cases $d = 2, 4, 12$ when there is an additional square, viz. H_3.

The second corollary contributes to the colorful palette of the results concerning triangular numbers.

Corollary 2. (Corollary of Lemma 5.) Let $\triangle_k$ denote the k^{th} triangular number, i.e. $\triangle_k = \frac{k(k+1)}{2}$, $k \geq 1$, $k \in \mathbb{N}$. Then the diophantine equation

$$\frac{\triangle_y}{\triangle_x} = 4^t \quad , \quad y \neq x \tag{13}$$

has no solution in natural numbers x, y and t.

3. Preliminaries

Lemma 1. Let $D_1 \in \mathbb{Z}$, $D_1 \neq 0$. If $|D_1| < 2^{96}$ and $2^n + D_1 = x^2$ has a solution (n, x) then

$$n < 18 + 2\log_2 |D_1| \quad . \tag{14}$$

Proof. This is Corollary 2 in [2] due to Beukers.

Lemma 2. Let p be an odd power of 2. Then for all $x \in \mathbb{Z}$

$$\left| \frac{x}{p^{0.5}} - 1 \right| > \frac{2^{-43.5}}{p^{0.9}} \quad . \tag{15}$$

Proof. We refer again to Beukers, [2].

Lemma 3. Let $D_2 \in \mathbb{N}$ be odd. The equation $2^n - D_2 = x^2$ has two or more solutions in positive integers n, x if and only if $D_2 = 7, 23$ or $2^k - 1$ for some $k \geq 4$. The solutions, in these exceptional cases, are given by the following table.

$D_2 = 7$	$(n, x) = (3, 1), (4, 3), (5, 5), (7, 11), (15, 181)$,
$D_2 = 23$	$(n, x) = (5, 3), (11, 45)$,
$D_2 = 2^k - 1$, $(k \geq 4)$	$(n, x) = (k, 1), (2k - 2, 2^{k-1} - 1)$.

Proof. See Theorem 2 in [2].

Lemma 4. All natural solutions (n, x) of the inequalities

(16) $$0 < |2^n - x^2| < 4$$

in positive integers n and x are given by $(n, x) \in \{(1,1), (2,1), (3,3), (1,2)\}$.

Proof. In virtue of Lemma 1, $n < 22$ and the verification of all possible values n gives the solutions above. ■

Lemma 5. Let t be an arbitrary positive integer. If x and y are integers satisfying

(17) $$y^2 - 1 = 2^{2t}\left(x^2 - 1\right) \quad , \quad y > 1 \, , \; x > 1 \quad ,$$

then $x = 2^{t-1}$ and $y = 2^{2t-1} - 1$ for $t > 1$.

Proof. It is easy to see that (17) is not solvable if $t = 1$. Suppose that $t > 1$, $y > 1$ and $x > 1$ satisfy (17). Then y is odd and

(18) $$\frac{y-1}{2} \cdot \frac{y+1}{2} = 2^{2t-2}\left(x^2 - 1\right) \quad .$$

The greatest common divisor of $\frac{y-1}{2}$ and $\frac{y+1}{2}$ is 1 and $2^{2t-2}(\geq 4)$ divides exactly one of the terms on the left hand side of (18). Consequently, $y = 2^{2t-1}k \pm 1$ with some integer $k \geq 1$. By (17) we have $y < 2^t x$, therefore $2^{t-1}k \leq x$. Moreover, it follows that

(19) $$2^{2t-2}k^2 - k = x^2 - 1 \qquad \text{or} \qquad 2^{2t-2}k^2 + k = x^2 - 1 \quad .$$

In the first case, clearly, $k = 1$ provides the solution $x = 2^{t-1}$, $y = 2^{2t-1} - 1$. If $k > 1$, then the inequalities

(20) $$x^2 = 2^{2t-2}k^2 - (k-1) < 2^{2t-2}k^2 \leq x^2$$

lead to contradiction.

In the second case of (19) it follows that

(21) $$\left(2^{t-1}k\right)^2 < \left(2^{t-1}k\right)^2 + k + 1 = x^2 < \left(2^{t-1}k + 1\right)^2 \quad ,$$

which is impossible. ■

Lemma 6. Let n, m and x be positive integers satisfying $2 \leq m < n$ and

(22) $$2^n + 2^m + 1 = x^2 \quad .$$

Then $x = 2^{m-1}(2k+1) \pm 1$ with some $k \in \mathbb{N}$.

Proof. Assume that (n, m, x) is a solution of (22) under the assumptions made. For $m = 2$ the lemma trivially states that x is odd. If $m \geq 3$, then the congruence

(23) $$x^2 \equiv 1 \pmod{2^m}$$

has exactly four incongruent solutions, namely $x \equiv 1$, $x \equiv 2^{m-1} - 1$, $x \equiv 2^{m-1} + 1$ and $x \equiv 2^m - 1 \pmod{2^m}$.

The first and fourth cases are impossible because, by (22), $x = 2^m l \pm 1$, $(l \in \mathbb{N}\, , \; l \geq 1)$ leads to

(24) $$2^{n-m} + 1 = 2^m l^2 \pm 2l \quad .$$

The second and third solutions of (23) provide

$$x = 2^{m-1}(2k+1) \pm 1 \quad , \quad (k \in \mathbb{N}) \quad . \tag{25}$$

■

Lemma 7. If n, m and x are natural numbers for which $m < n$ and $n < 2m - 2$, then

$$2^n + 2^m + 1 = x^2 \tag{26}$$

implies $(n, m, x) = (5, 4, 7)$.

Proof. The conditions of the lemma give $m \geq 4$. Suppose that (n, m, x) satisfy (26). Combining Lemma 6 and (26) we obtain

$$2^n + 2^m + 1 = r^2 2^{2m-2} \pm r 2^m + 1 \quad , \tag{27}$$

where r is a positive odd integer. Since $2m - 2 \geq n + 1$, we get

$$2^m(1 \mp r) \geq (2r^2 - 1)2^n \quad . \tag{28}$$

Hence $r = 1$, $x = 2^{m-1} - 1$ and $n \leq m + 1$. By $m < n$ we have $n = m + 1$ and we can conclude that $m = 4$, $n = 5$ and $x = 7$. ■

Lemma 8. If D, k and x are positive integers, $k \geq 3$ and

$$2^{D+8k} + 2^{4k} + 1 = x^2 \quad , \tag{29}$$

then $D > 56k - 32$.

Proof.

Let $\nu_2(n)$ denote the 2-adic value of the integer n . Assume that the integers D, k and x satisfy the conditions of the lemma. By Lemma 6 we have two possibilities for x.

A) First consider the case $x = 2^{4k-1}(2u_0 + 1) + 1$, $(u_0 \geq 0)$. By (29) we obtain

$$2^{D+4k-1} = 2^{4k-3}(2u_0 + 1)^2 + u_0 \quad , \tag{30}$$

where u_0 must be positive and $u_0 = 2^{4k-3}u_1$ with some positive odd integer u_1. Otherwise, dividing (30) by $2^{\min\{4k-3, \nu_2(u_0)\}}$, it leads to contradition. In the sequel, this type of argument will be applied without any further notice. It follows that

$$2^{D+2} = 2^{8k-4}u_1^2 + 2^{4k-1}u_1 + (u_1 + 1) \quad . \tag{31}$$

Then $u_1 + 1 = 2^{4k-1}u_2$ for some suitable positive odd integer u_2, and by (31) we get

$$2^{D-4k+3} = 2^{4k-3}\left(2^{4k-1}u_2 - 1\right)^2 + 2^{4k-1}u_2 + (u_2 - 1) \quad . \tag{32}$$

Clearly, $u_2 \neq 1$, $u_2 - 1 = 2^{4k-3}u_3$, $(u_3 \in \mathbb{N}, u_3 \equiv 1 \pmod 2)$, further

$$2^{D-8k+6} = 2^{8k-2}\left(2^{4k-3}u_3 + 1\right)^2 - 2^{4k}\left(2^{4k-3}u_3 + 1\right) + 2^{4k-1}u_3 + (u_3 + 5) \quad . \tag{33}$$

It is easy to see that $u_3 + 5 = 2^{4k-1}u_4$, where u_4 is an odd natural number. Hence

$$
\begin{aligned}
2^{D-12k+7} &= 2^{4k-1}\left(2^{4k-3}\left(2^{4k-1}u_4-5\right)+1\right)^2 - \\
&\quad -2^{4k-2}\left(2^{4k-1}u_4-5\right)+2^{4k-1}u_4+(u_4-7) \quad . \qquad (34)
\end{aligned}
$$

By (34) we conclude that $u_4-7=2^{4k-2}u_5$. Here the odd integer $u_5=\frac{u_4-7}{2^{4k-2}}$ is positive because $k\geq 3$ and $u_4>0$. It follows that

$$
\begin{aligned}
2^{D-16k+9} &= 2^{8k-5}\left(2^{4k-1}\left(2^{4k-2}u_5+7\right)-5\right)^2+2^{8k-2}\left(2^{4k-2}u_5+7\right) - \\
&\quad -2^{8k-3}u_5+(u_5-12)2^{4k-1}+(u_5+21) \quad . \qquad (35)
\end{aligned}
$$

Finally, $u_5+21=2^{4k-1}u_6$, $(u_6\in\mathbb{N}, u_6\equiv 1 \pmod 2)$, and then $u_6-33=2^{4k-4}u_7$ leads to the equality

$$
2^{D-24k+14}=\left(2^{4k-1}\left(2^{4k-2}Q_1+7\right)-5\right)^2+R_1+S_1 \quad , \qquad (36)
$$

where

$$
Q_1=2^{4k-1}\left(2^{4k-4}u_7+33\right)-21 \quad , \quad R_1=2^3\left(2^{4k-2}Q_1+7\right)-2^2Q_1 \quad , \qquad (37)
$$

$$
S_1=2^3\left(2^{4k-4}u_7+33\right)+u_7 \quad , \quad u_7\in\mathbb{N}\ ,\ u_7\equiv 1 \pmod 2 \ . \qquad (38)
$$

Obviously, $Q_1>2^{8k-5}$, $S_1>0$, $R_1>0$. Therefore $2^{D-24k+14}>2^{32k-16}$ and

$$
D>56k-30 \quad . \qquad (39)
$$

B) In the second case replace x by $2^{4k-1}(2u_0+1)-1$, $(u_0\geq 0)$ in (29) and, similarly as above, the substitutions $u_0=2^{4k-3}u_1-1$, $u_1=2^{4k-1}u_2+1$, $u_2=2^{4k-3}u_3-1$, $u_3=2^{4k-1}u_4+5$, $u_4=2^{4k-2}u_5-7$, $u_5=2^{4k-1}u_6+21$ and $u_6=2^{4k-4}u_7-33$ lead to the equality

$$
2^{D-24k+14}=\left(2^{4k-1}Q_2+5\right)^2-8Q_2+R_2 \quad , \qquad (40)
$$

where

$$
Q_2=2^{4k-2}\left(2^{4k-1}\left(2^{4k-4}u_7-33\right)+21\right)-7 \quad , \qquad (41)
$$

$$
R_1=2^2\left(2^{4k-1}\left(2^{4k-4}u_7-33\right)+21\right)-2^3\left(2^{4k-4}u_7-33\right)-u_7 \quad , \qquad (42)
$$

and $u_7\in\mathbb{N}$, $u_7\equiv 1 \pmod 2$. It can be proved that $Q_2>2^{12k-8}$, $R_2>0$ and we have $2^{D-24k+14}>2^{32k-18}$, which, together with (39), implies $D>56k-32$. The proof of Lemma 8 is complete. ■

Lemma 9. If a and c are non-negative integers satisfying

$$
a^2+(a+1)^2=c^2 \quad , \qquad (43)
$$

then $a=2P_nP_{n+1}$, $a+1=P_{n+1}^2-P_n^2$ or conversely, where P_k denotes the k^{th} term of the Pell sequence defined by $P_0=0$, $P_1=1$ and $P_k=2P_{k-1}+P_{k-2}$, $(k\geq 2)$.

Proof. Probably this is an old result. For the proof see, for instance, Cohn [3]. ■

4. Proofs

Proof of Theorem 1. Obviously, each element of the set

$$T = \{(n,m) \in \mathbb{N}^2 \mid n = 2t, m = t+1, t \in \mathbb{N}, t \geq 1\} \tag{44}$$

(with some suitable $x \in \mathbb{N}$) satisfies the relations

$$2^n + 2^m + 1 = x^2 \quad , \quad n \geq m \geq 1 \quad . \tag{45}$$

Let S denote the set of solutions (n,m) of (45), further let $M_1 = (5,4)$ and $M_2 = (9,4)$. We have to show that the set of exceptional solutions is $S \setminus T = \{M_1, M_2\}$.

Observe that $2^{n-m} + 1 = \frac{x^2-1}{2^m} \in \mathbb{N}$ if $(n,m) \in S$, further

$$2^{2n-2m} + 2^{n-m+1} + 1 = \left(\frac{x^2-1}{2^m}\right)^2 \quad . \tag{46}$$

Hence a solution (n,m) of (45) provides $(2n-2m, n-m+1) \in S$, except when $2n - 2m < n - m + 1$, i.e. $n = m$. But Lemma 4 implies that the only solution (n,m) with $n = m$ is $(2,2) \in T$.

In the sequel, we assume that $n > m$. Then the transformation

$$\tau \ : \ (n,m) \longmapsto (2n-2m, n-m+1) \quad , \quad (n > m) \tag{47}$$

induces a map of $S \setminus \{(2,2)\}$ into S.

Figure 1. Map τ on the solutions of the equation $2^n + 2^m + 1 = x^2$

The map τ has important properties. If $(n,m) \in S$, then let $\delta(n,m)$ denote the distance $n - m$ of the exponents n and m.

Property 1. $\delta(\tau(n,m)) = \delta(n,m) - 1$. In particular, $\tau(n,m) \neq (n,m)$, i.e. the map has no fixed points.

Property 2. If $(n,m) \in T \setminus \{(2,2)\}$, more precisely if $(n,m) = (2t, t+1)$, $t \geq 2$, then $\tau(n,m) = (2(t-1), t) \in T$ is the '*lower neighbour*' solution of (n,m) in T. Thus the elements of the set T are ordered by τ. Moreover $\delta(\tau(2t, t+1)) = t - 2$, $(t \geq 2)$ shows that all natural numbers occur as a difference of the exponents in the solution of (45).

Property 3. If (n,m) is an exceptional solution (i.e. $(n,m) \in S \setminus T$), then $\tau(n,m) \in T$ since $\tau(n,m) = (2\delta(n,m), \delta(n,m)+1)$. Especially, $\tau(5,4) = (2,2)$, $\tau(9,4) = (10,6)$.

If $m = 1$, then Lemma 4 implies a solution with $n < m$, which contradicts the assertion $n > m$.

Now suppose that the integers n and m satisfy $2 \leq m < n$ and (45). Reconsidering the map

$$\tau \ : \ S \setminus \{(2,2)\} \longmapsto S \tag{48}$$

with (47), by Properties 1-3 we have to prove that there are exactly two cases when $(n,m) \neq (n_1,m_1)$ and $\tau(n,m) = \tau(n_1,m_1)$. In other words, we must show that the system of the equations

$$2^n + 2^m + 1 = x^2 \tag{49}$$
$$2^{n+d} + 2^{m+d} + 1 = y^2 \tag{50}$$

in positive integers n, m, d, x, y with $2 \leq m < n$ has exactly two solutions.

Taking such a solution, obviously both $x > 1$ and $y > 1$ are odd. It follows from (49) and (50) that

$$y^2 - 1 = 2^d \left(x^2 - 1\right) \quad , \tag{51}$$

and by Lemma 5 we infer that d must be odd.

Observe that one of (n,m) and $(n+d,m+d)$ has to belong to the set $T \setminus \{(2,2)\}$. On the contrary, if both (n,m) and $(n+d,m+d)$ are exceptional, by the properties of the transformation τ there exists a solution $(n_2,m_2) \in T \setminus \{(2,2)\}$ such that $\tau(n_2,m_2) = \tau(n,m) = \tau(n+d,m+d)$. But in this case one of the distances $|n_2 - n| = |m_2 - m|$ and $|n_2 - (n+d)| = |m_2 - (m+d)|$ has to be even since d is odd, which contradicts again to Lemma 5. Therefore, we distinguish two cases.

A) First let (50) be the exceptional case, consequently $(n,m) \in T \setminus \{(2,2)\}$, and by (44) it follows that $n = 2m - 2$, which, together with (50), implies

$$2^{2m-2+d} + 2^{m+d} + 1 = y^2 \quad . \tag{52}$$

Here, if $m \geq 3$, then the exponents $m+d$ and $2m-2+d$ on the left hand side satisfy the conditions of Lemma 7. Thus we conclude that $m = 3$, $d = 1$, $y = 7$ is the only solution of (52) (and $n = 4$, $x = 5$ of (49)). It gives $M_1 \in S$. On the other hand, if $m = 2$, then $n = 2m - 2 \leq m$ leads to contradiction.

B) The second possibility is that (49) is the exceptional case, while $(n+d,m+d) \in T \setminus \{(2,2)\}$, i.e. $n = 2m + d - 2$. Then by (49) we have

$$2^{2m+d-2} + 2^m + 1 = x^2 \quad . \tag{53}$$

It is easy to show that one of the exponents must be even in (53). Since d is odd, therefore m has to be even. Put $m = 2r$, where $r \in \mathbb{N}$, $r \geq 1$, and let $D = d - 2$. If $D = -1$, then (53) is equivalent to

$$2^{4r-1} + 2^{2r} + 1 = x^2 \quad . \tag{54}$$

Observe that the left hand side of (54) is a sum of $\left(2^{2r-1}\right)^2$ and $\left(2^{2r-1}+1\right)^2$, hence $a = 2^{2r-1}$, $a+1$ and x form a Pythagorean triple. Since a is even, by Lemma 9 we have $2^{2r-1} = 2P_nP_{n+1}$ with some $n \in \mathbb{N}$. Therefore, both P_n and P_{n+1} are power of 2, which is impossible if $n \geq 2$ because P_n and P_{n+1} are coprime. Since $P_0 = 0$, the only possibility is $n = 1$, but $1 \cdot 2 \neq 2^{2r-2}$.

Consequently, $D \geq 1$ and we have

$$2^{D+4r} + 2^{2r} + 1 = x^2 \quad . \tag{55}$$

The left hand side of (55) is quadratic residue $\pmod 5$ if and only if r is even. Put $r = 2k$, ($k \in \mathbb{N}$, $k \geq 1$). Thus

$$2^{D+8k} + 2^{4k} + 1 = x^2 \quad , \tag{56}$$

which is equivalent to

$$\frac{x}{2^{\frac{D+8k}{2}}} - 1 = \frac{2^{4k}+1}{2^{\frac{D+8k}{2}}\left(x + 2^{\frac{D+8k}{2}}\right)} \quad . \tag{57}$$

Applying Lemma 2 to the left hand side of (57), and using that (56) gives $2^{\frac{D+8k}{2}} < x$, we obtain

$$\frac{2^{-43.5}}{2^{(D+8k)\cdot 0.9}} < \frac{2^{4k}+1}{2\cdot 2^{D+8k}} \quad . \tag{58}$$

We see that $2^{4k}+1 < 2^{4k+0.5}$ if $k \geq 1$, and by (58) it follows that

$$D < 32k + 430 \quad . \tag{59}$$

On the other hand, considering (56), Lemma 8 provides $D > 56k - 32$, which, together with (59) implies $k \leq 19$. Finally, applying Lemma 1 to (56) with $D_1 = 2^{4k}+1$, $(k \leq 19)$ we conclude that $D \leq 19$, too. A simple computer search shows that equation (56) with odd $D \leq 19$ and $k \leq 19$ has only one solution $D = 1$, $k = 1$, $x = 23$. Hence we obtain the third exceptional solution of (45): $(n,m) = (D+8k, 4k) = (9,4)$, and there are no others. So the proof of Theorem 1 is complete. ■

Proof of Theorem 2. Suppose that $(n,m,x) \in \mathbb{N}^3$ is a positive solution of the diophantine equation

$$2^n - 2^m + 1 = x^2 \quad . \tag{60}$$

Consider the case $n \geq m$. First let $m \geq 4$. Then (60) is equivalent to the equation

$$2^n - D_2 = x^2 \quad , \tag{61}$$

where the positive number $D_2 = 2^m - 1$ is odd. By Lemma 3, we find

$$(n,x) = (m,1)\ ,\ (2m-2, 2^{m-1}-1) \tag{62}$$

as the set of all the solutions of (61) with $m \geq 4$. This result leads to the following solutions of (60):

$$(n,m,x) = (t,t,1) \quad , \quad t \in \mathbb{N}\ ,\ t \geq 4 \quad ; \tag{63}$$

$$(n,m,x) = (2t, t+1, 2^t - 1) \quad , \quad t \in \mathbb{N}\ ,\ t \geq 3 \quad . \tag{64}$$

The famous case $m = 3$ of (60) has five solutions given by the table in Lemma 3. Among them $(n,m,x) = (3,3,1)$ can be joined to the set (63) with the parameter $t = 3$, moreover, $(n,m,x) = (4,3,3)$ to the set (64) with $t = 2$.

If $m = 2$ or $m = 1$, then Lemma 4 gives the result $(n,m,x) = (2,2,1)$ or $(n,m,x) = (1,1,1)$, respectively. These triplets may be added, for example, to (63) with $t = 2$ and with $t = 1$, respectively.

Finally, it is easy to see that (60) has no solution with $0 < n < m$. Avoiding the repetitions we may summarise the results above as Theorem 2 states. ■

Proof of Theorem 3. Assume that $(n,m,x) \in \mathbb{N}^3$ with $n \geq m > 0$ is a solution of the equation

$$2^n + 2^m - 1 = x^2 \quad . \tag{65}$$

If $m \geq 2$, then $2^n + 2^m - 1$ is a quadratic non-residue modulo 4; if $m = 1$, then apply Lemma 4 to have $(n,m,x) = (3,1,3)$.

Now suppose that $(n, m, x) \in \mathbb{N}^3$ is a solution of the equation

$$2^n - 2^m - 1 = x^2 \quad . \tag{66}$$

Clearly, $n > m$ and $m < 2$. For $m = 1$ apply Lemma 4 to prove the statement. ■

Proof of Corollary 1. Both sequences G and H have companion polynomial $c(x) = x^2 - 3x + 2$ with zeros $x = 2$ and $x = 1$. It is well known that the terms G_m (and H_m) can be expressed in explicit form. Here by $a_G = G_1 - G_0 = 2^d + 1$ ($a_H = H_1 - H_0 = 2^d - 1$) and by $b_G = -G_1 + 2G_0 = 1$ ($b_H = -H_1 + 2H_0 = 1$) we have

$$G_m = a_G 2^m + b_G = 2^{m+d} + 2^m + 1 \quad , \tag{67}$$

$$H_m = a_H 2^m + b_H = 2^{m+d} - 2^m + 1 \quad . \tag{68}$$

Thus to determine all the squares in the recurrences G and H is equivalent to solve the equations (6) and (7) with $n = m + d$ (i.e. $n \geq m$). ■

Proof of Corollary 2. $\triangle_y = 4^t \triangle_x$ ($y \neq x$, $y > 0$, $x > 0$) implies

$$y_1^2 - 1 = 4^t \left(x_1^2 - 1\right) \quad , \tag{69}$$

where $y_1 = 2y + 1 \geq 3$ and $x_1 = 2x + 1 \geq 3$. In virtue of Lemma 5, (69) has no solution under the given conditions. ■

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